Modeling Rainfall Runoff Responses and Antecedent Moisture Effects Using Principles of System Identification

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Abstract

Rainfall-runoff dynamics of surface water, combined sewer, and separate sewer systems can be highly impacted by antecedent moisture conditions, i.e. the relative wetness or dryness of the system. Simulating these dynamics with a reasonable degree of accuracy is critical so that predictive models of systems can be developed for risk management, planning, and design of upgrades. This paper presents the results of 25 years of work formulating, applying and refining a hydrologic model using the principles of system identification from the field of aerospace control systems to find the simplest mathematical model that accurately describes the relationship between system inputs and the flow output. The model addresses the impacts of antecedent moisture conditions on the rainfall-runoff process. The development, process and equations of the model are presented. Developing and testing the model was done primarily from observations in the Midwest U.S. where both preceding rainfall and seasonal hydrologic conditions impact antecedent moisture dynamics. For these systems, the model described here is perhaps the most parsimonious that can accurately simulate these dynamics. This provides several advantages to the modeler, including ease of use, fewer parameters to calibrate, ability to quickly identify optimal parameters, and ease to represent in a numerical computer routine. Physical interpretation of the model structure and parameters is possible, providing the modeler with useful insights into the physical processes driving the rainfall-runoff dynamics.

Useful resources:

- Antecedent Moisture Effects on Rainfall-Runoff Systems (video).
- Overview of the AMM modeling process (article).
- How to use the AMM (article).
- AMM Equations Companion Spreadsheet (Excel file)
- Tutorial Videos on AMM (index of videos).

1 Introduction

1.1 Background

Rainfall-runoff systems are prevalent throughout the natural and built environment, and include surface runoff, stormwater systems, combined sewer systems, and separate sanitary sewer systems.

Modeling of these systems is performed for applications like flood control, combined sewer overflow control, and separate sanitary sewer overflow control. Billions of dollars in capital improvements are designed each year based on the outcomes and accuracy of models. The purpose of a model is to simulate unobserved conditions from a mathematical description of the system based on past system performance. In that respect, for a model to be useful, it must be capable of making accurate predictions of future events.

Antecedent moisture conditions, or the relative wetness or dryness of a system, can have a tremendous impact on rainfall-runoff dynamics. The magnitude of the runoff (flow) from rainfall can be affected by how wet the drainage area is from prior conditions. Wetter conditions can produce more runoff, and drier conditions can produce less runoff. Wetness conditions can be affected by a multitude of hydrologic conditions that include items such as prior rainfall, depression storage, air temperature, evaporation, evapotranspiration, solar radiation, soil types, and many other factors. Hydrologic systems can exhibit a wide variation in their response to antecedent moisture conditions. The impact could be as small as initial depression storage on an impervious surface that only affects runoff by a small percentage, or it could be as large as varying wetness conditions in a separate sanitary sewer system that change inflow and infiltration volumes by an order of magnate or more.

Understanding the relative impact of antecedent moisture on these systems is critical for engineering design and system operations. These impacts can be critically important for developing accurate predictive models of systems that are sensitive to antecedent moisture conditions. The importance of accurately accounting for antecedent moisture effects has been covered extensively in the literature (Hogan 2000 et al., Van Pelt 2002 et al., Czachorski et al., 2006, Czachorski et al., 2011, Cheng et al., 2011, and others).

Event based models, in general, provide a single set of variables that are fixed in time for the definition of a unit hydrograph, from which a flow response is generated. Such models can be calibrated to system responses from individual storm events with a reasonable degree of accuracy (e.g. RTK or other unit hydrograph methods). As such, these models tend to implicitly incorporate the wetness conditions present during that storm only. Unless these unit hydrograph methods allow for time-based variation of variables, especially for systems impacted by antecedent moisture conditions, such models may not be very useful for making reasonably accurate predictions of other events than those they were calibrated to. Simulating the rainfall-runoff dynamics of such systems requires the use of an accurate, continuous model that is developed to simulate many storms that may occur over a wide range of antecedent moisture conditions. Compared to single event simulation, a continuous simulation that allows for variation of modeling parameters over time due to antecedent moisture effects can more correctly represent antecedent conditions by incorporating processes of both dry weather periods and wet weather periods (Cheng et al., 2011).

1.2 The System Identification Approach

System identification is a methodology for building mathematical models of dynamic systems using observed data from the system, and thus is part of the basic scientific methodology (Ljung 1998). System identification is used for developing models for control systems in the aerospace fields. Common system identification approaches start from measurements of system inputs and outputs, and then attempt to identify a mathematical relationship between them. The simplest model structure with the fewest number of parameters is sought, because over-parameterizing a model typically yields less accuracy in its ability to predict (Burnham et al., 2002).

The system identification approach is an alternative to a white-box model based on first principles of the physical processes in the system (such as conservation of mass, energy, momentum, etc.) Developing models based on first principles can be overly complex and even impossible due to the complex nature of some systems and processes (Toth 2010). There are fundamental problems in the application of physically-based models for prediction in hydrology due to limitations of the model equations to represent the complex physical reality, grid size requirements, and parameter details necessary at the scale required (Beven 1989). It is likely that a correct representation incorporating all of the pertinent physical parameters would lead to a mathematical description that would be highly uncertain due to the accumulation of individual parameter uncertainty, and such a model would require extraordinarily precise knowledge of sensitive parameters in order to validate the accuracy of the model as a whole (Van Pelt et al., 2002). Accurate representation of the physical process of inflow and infiltration into sanitary sewers is difficult due to the complex pathways that occur (Vallabhaneni et al., 2002). These factors are greatly compounded when attempting to simulate the impacts of antecedent moisture conditions on these systems.

Whereas system identification models can be developed without focusing on the details of what is happening inside the system. The resulting system identification models can be black box or grey box models. In a black box model, the model is identified mathematically only from the measured inputs and outputs without consideration of the internal workings of the system. In a grey box model, portions of the model structure are selected at the start based on insights into the system, and therefore physical interpretations of the model can be made. The model presented here is a grey box model.

1.3 The Parsimony Principle

Parsimony is the concept that a model should be as simple as possible with respect to the included variables, model structure, and number of parameters (Burnham et al., 2002). This is also known as Occam's Razor, which is the notion that entities should not be multiplied unnecessarily (Gibbs et al., 1997). Developing a model using a system identification approach requires a balance between reducing the number of parameters (parsimony) and constructing a model that has a good fit to observed data (model accuracy).

Parsimony lies between the evils of underfitting and overfitting (Forster et al., 1994). A simple rule was developed by James (2005) for determining optimal modeling complexity: "In determining the best level of complexity, test simple models first, proceeding to more complex, until the required accuracy of the computed response function is achieved. Use the least number of processes, discretized spaces, and the biggest time step that delivers the required uncertainty". This is also known as the simplicity principle, which states that for the same level of predictive accuracy, the simplest model should be selected (Yan 1990). Relying on the principle of parsimony, a predictive runoff model can be created that is generally more accurate while having significantly fewer variables to track, edit, and manage (Barlock et al., 2016).

1.4 Purpose and Scope

The characteristics of the system identification approach and parsimony are well suited for developing models to simulate rainfall runoff responses and antecedent moisture effects. These principles have been applied to develop the model for simulating the rainfall runoff responses and antecedent moisture effects described in this paper. Until now, the details of the model have been held as a trade secret. With this paper, the equations and model process are released into the public domain.

Several papers have been published over the years on the application and performance of the model (Czachorski et al., 2001, Van Pelt et al., 2002, Czachorski et al., 2005, Czachorski et al., 2006, Kuhns et al., 2007, Ricker et al., 2008, Kuhns et al., 2010, Czachorski et al., 2011, Kuehlmann et al., 2013, Czachorski et al., 2014), so those findings are not repeated here. The current paper is focused on presenting the development, process and equations of the model.

2 The Antecedent Moisture Model Development

2.1 Process

A system identification approach was applied to the rainfall-runoff process for the purpose of developing a simple model that is capable of accurately simulating and predicting the impacts of antecedent moisture effects. The model was developed through the following process:

- Examining a vast amount of input-output data from various systems to understand the dynamics present in the rainfall-runoff process from antecedent moisture effects.
- Tabulating a list of the potential input observations that may impact the measured flow output. Initially, these included several potential precipitation and climatological inputs.
- Identifying candidate input observations and a potential model structure that describes the mathematical relationship between the inputs and the flow output.
- Testing the potential model structure against a vast amount of input-output observations to identify areas for improvement and simplification to the model parameterization and structure.
- Refining the model by seeking the simplest model structure, fewest input observations, and fewest model parameters that sufficiently represents the input-output relationship.
- Applying the resultant model to numerous systems over the years to validate its ability to accurately simulate the rainfall runoff responses and antecedent moisture effects.

This methodology was initially applied in the late 1990’s to develop a system identification approach to simulating antecedent moisture effects (Czachorski et al., 2001). Since the initial development, the Antecedent Moisture Model (AMM) has been applied to hundreds of systems and refined over the years to simplify and improve the model. The results of this process are a series of equations outlined in this paper that provide a mathematical description of antecedent moisture effects on the rainfall-runoff process.

2.2 Parameterization

The AMM was developed primarily from observations in the Midwest U.S. where both preceding rainfall and seasonal effects, represented by variations in air temperature in the model, impact antecedent moisture conditions. For these systems, the AMM described here is perhaps the most parsimonious model that can accurately simulate these dynamics. This provides several advantages to the modeler, including ease of use, fewer parameters to calibrate, ability to quickly identify optimal parameters, and ease to represent in a numerical computer routine. The final model contains only five (5) primary parameters, as follows:

1. A Shape Factor that describes the hydrographs recession.
2. A Response Factor affine constant that describes how rainfall increases runoff.
3. An Antecedent Moisture Retention Factor that describes the recession of antecedent moisture levels.
4. A High Temperature Factor that describes how rainfall increases antecedent moisture levels during high temperatures.

5. A Low Temperature Factor that describes how rainfall increases antecedent moisture levels during low temperatures.

As described in this paper, other settings are used in the model for averaging, smoothing or delaying the temperature and precipitation inputs. For example, the temperature input time series is typically averaged over the previous 10 or 20 days to prevent short term temperature variations from impacting long-term antecedent moisture levels in the model. These settings typically don't vary much from system to system, and thus are not primarily parameters for calibrating the AMM.

2.3 Application

Different input observations and a different model structure may be necessary to accurately simulate antecedent moisture dynamics in different climatologic regions. For example, in a temperate region such as Southern California, where seasonal effects are mild, it may be possible to simulate antecedent moisture effects from only preceding rainfall, without the seasonal variations. In this case, only four model parameters would be used by combining the High and Low Temperature Factors into a single constant parameter. Other regions of the world may have other model structures and parameterizations that work best. Choosing the best model for an application is one of the most difficult aspects of the modeling process and has a major effect on the success of the project (James 2005). The author invites others to apply the system identification approach to their models by examining the equations presented here, and modifying their structure, inputs and parameterization to identify the simplest and most accurate model for their application and system.

3 Antecedent Moisture Model Equations

3.1 Overview

This paper outlines the equations for modeling rainfall runoff responses and antecedent moisture effects developed using the principles of system identification described above. The equations are described in a step-by-step process to show their development and build-up, and consist of the following major components:

- Linear rainfall-runoff transfer function that simulates flows from rainfall without antecedent moisture effects.
- Antecedent moisture retention transfer function that simulates the impacts of preceding rainfall on antecedent moisture levels.
- Temperature factor transfer function that simulates the impact of seasonal effects on antecedent moisture levels using air temperature.
- Affine constant that has the effect of placing a floor or lower bounds to the antecedent moisture level.
- Base flow transfer Function that simulates the base flow component or Ground Water Infiltration (GWI) that varies very slowly with a very long response time that can take weeks or months to react to climatological variations like precipitation and air temperature.
- Precipitation smoothing and delay functions that average and delay the precipitation time series input so that travel time delays and hydrograph smoothing can be done.
- Moving average temperature function that is an averaging function to smooth the temperature time series input so that short term temperature variations do not impact long-term variations in antecedent moisture levels.

3.2 Linear Rainfall-Runoff Transfer Function

A transfer function is a mathematical function which theoretically models a system's output from possible inputs (Parr 1993). This allows the establishment of an equation between the inputs and outputs of a physical system, without necessitating an understanding of the detailed, physical processes governing the relationship between the input and output. A simple linear transfer function can be used to simulate the rainfall-runoff response. The transfer function is represented as a linear difference equation to compute flow at discrete time intervals from discrete measurements of precipitation. Equation 1 shows the transfer function, which has a single input of a rainfall time series, and single output of a flow time series, and two parameters. This is a deterministic, lumped, conceptual model for the rainfall-runoff process.

$$Q_t = RF * P_t + SF * Q_{t-1} \quad (1)$$

where:

Q_t = Flow rate output at the current time step, *t*
Q_{t-1} = Flow rate at the previous time step, *t-1*
P_t = Precipitation input at the current time step, *t*
RF = Constant Response Factor
SF = Constant Shape Factor of hydrograph, bound by [0-1]

Some notes on the linear transfer function:

- The output from Equation 1 is a hydrograph time series of how the flow varies in time in response to precipitation as an input variable.
- A larger Response Factor *(RF)* yields a larger hydrograph. Because the rainfall-runoff transfer function is linear, superposition applies, and the output hydrograph will scale proportional to the Response Factor. For example, doubling the Response Factor will double all points of the output hydrograph.
- A larger Shape Factor *(SF)* yields a longer tail on the hydrograph. The Shape Factor is a decay parameter strictly bound by [0-1] and causes the hydrograph to decay and approach zero after the precipitation input has stopped.
- Depending on the system being modeled and the level of accuracy desired, several rainfall-runoff transfer functions could be used to represent different components of the flow (impervious, pervious, inflow, infiltration, etc.) These can then be added together to represent the total flow, as described in later sections.
- Travel time delays can be accounted for by adjusting the rainfall input to P_{t-n}, where n represents the number of time steps to delay the rainfall input.
- By setting *RF + SF = 1* in Equation 1, the rainfall-runoff linear transfer function has the same form as the equation to represent an instantaneous unit hydrograph (IUH) (Bedient et al., 2019).

Figure 1 shows an example of the flow output from Equation 1. The transfer function is unitless and any units can be used for the input and output, provided they are used consistently.

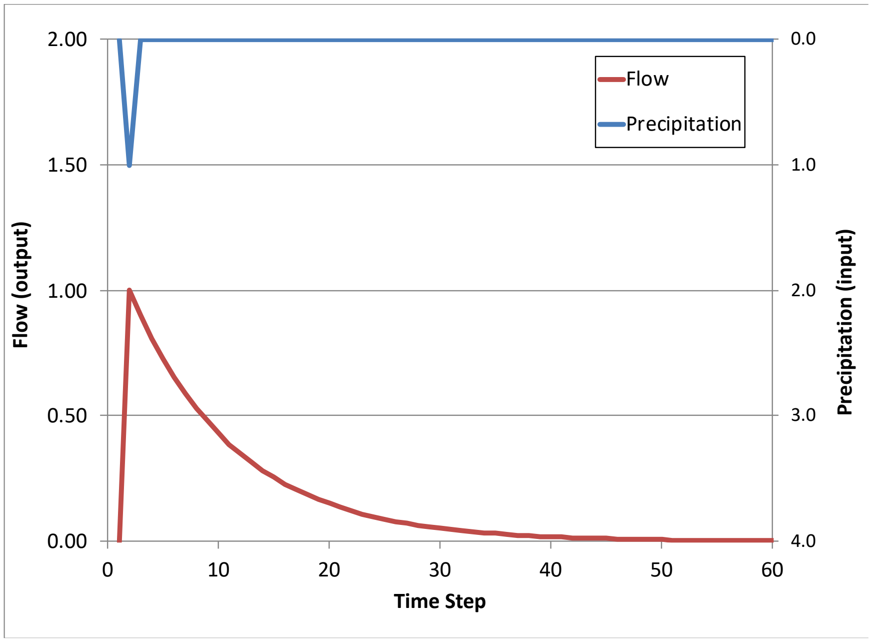

Figure 1 Flow output from the linear rainfall-runoff transfer function with a precipitation input of 1 at a single time step. SF = 0.9, RF = 1.

3.3 Antecedent Moisture Retention Transfer Function

Antecedent moisture conditions can impact the system response continuously during storm events and in between storm events. Application to many systems has shown that antecedent moisture conditions do not affect the Shape Factor *(SF)* in Equation 2, suggesting that this might be a characteristic of the physical system (such as district shape, slope, travel time, etc.), as opposed to temporal system changes like rainfall. This is very convenient for modeling antecedent moisture conditions, because variations in antecedent moisture can then be represented explicitly by changes, in time, to only the Response Factor (RF). Such a continuously varying the Response Factor *(RF)* is shown in Equation 2. In this equation, unlike Equation 1, the value of the Response Factor *(RF)* changes with time.

$$Q_t = RF_t * P_t + SF * Q_{t-1} \qquad (2)$$

where:

RF_t = Varying Response Factor at current time step, t

The continuously varying Response Factor (RF_t) from Equation 2 can be represented as a simple linear transfer function, as shown in Equation 3. This equation has a single input of a precipitation time series, and single output of the varying response factor, and two parameters.

$$RF_t = TF * P_t + AMRF * RF_{t-1} \qquad (3)$$

where:

RF_t = Varying Response Factor output at the current time step, *t*
RF_{t-1} = Response Factor at the previous time step, *t-1*
P_t = Precipitation at the current time step, *t*
TF = Constant Temperature Factor

AMRF = Constant Antecedent Moisture Retention Factor, bound by [0-1]

Some notes on the Antecedent Moisture Retention transfer function:

- The output from Equation 3 is a time series of how the Reponses Factor varies in time. It is similar in form to the hydrograph output from Equation 1, but the response factor time series represents how antecedent moisture conditions vary over time and affect the hydrograph response to rainfall.
- Temperature Factor (TF) is a constant in Equation 3 to describe the dynamics of antecedent moisture retention only due to precipitation. In the next section, the Temperature Factor will be allowed to vary continuously with air temperature hence the naming of this equation component.
- A larger Temperature Factor (TF) yields a larger response factor output time series. This parameter can be conceptualized as the degree to which precipitation increases antecedent moisture conditions. Because the antecedent moisture retention transfer function is linear, superposition applies, and the response factor output time series will scale proportional to the Temperature Factor. For example, doubling the Temperature Factor will double all points of the response factor output time series.
- A larger Antecedent Moisture Retention Factor (AMRF) yields a longer tail on the response factor output time series. This parameter can be conceptualized as the memory for how long antecedent moisture conditions are affected after precipitation. The Antecedent Moisture Retention Factor is a decay parameter strictly bound by [0-1] and causes the response factor output time series to decay and approach zero after the precipitation input has stopped.
- Continuously varying the Response Factor in Equation 2 based on the output from Equation 3 makes Equation 2 nonlinear in parameters.
- When modeling several different components of the flow (impervious, pervious, inflow, infiltration, etc.), each may be affected by their own antecedent moisture retention transfer function.
- Note that Equation 2 defaults to Equation 1 if the Response Factor (RF_t) was a constant in time.

Figure 2 shows an example flow output from Equation 1 with a constant Response Factor with two back to back precipitation events. Figure 3 shows an example Response Factor and flow output from Equations 2 and 3 with a continuously varying Response Factor with the same precipitation as Figure 2. The continuously varying Response Factor shown in Figure 3 results in a higher flow output for both storms compared to the linear rainfall-runoff transfer function shown in Figure 2.

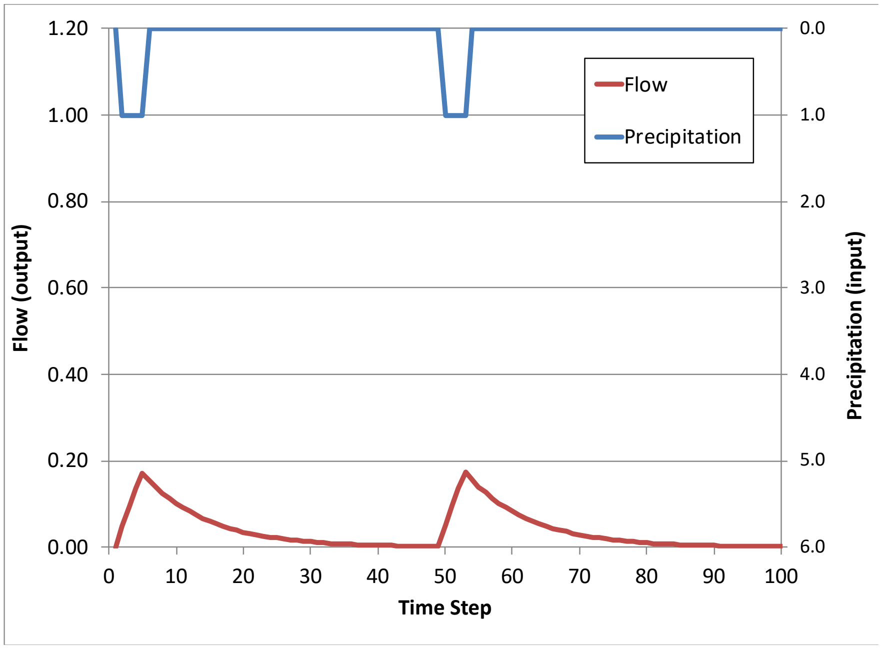

Figure 2 Linear rainfall-runoff output from Equation 1 with back-to-back storms. SF = 0.9, RF = 0.05. Precipitation inputs of a value of 1 in the first four time steps, and four more precipitation inputs of a value of 1 starting in the 50th time step. The linear rainfall-runoff transfer function yields the same output for both storms with a constant Response Factor.

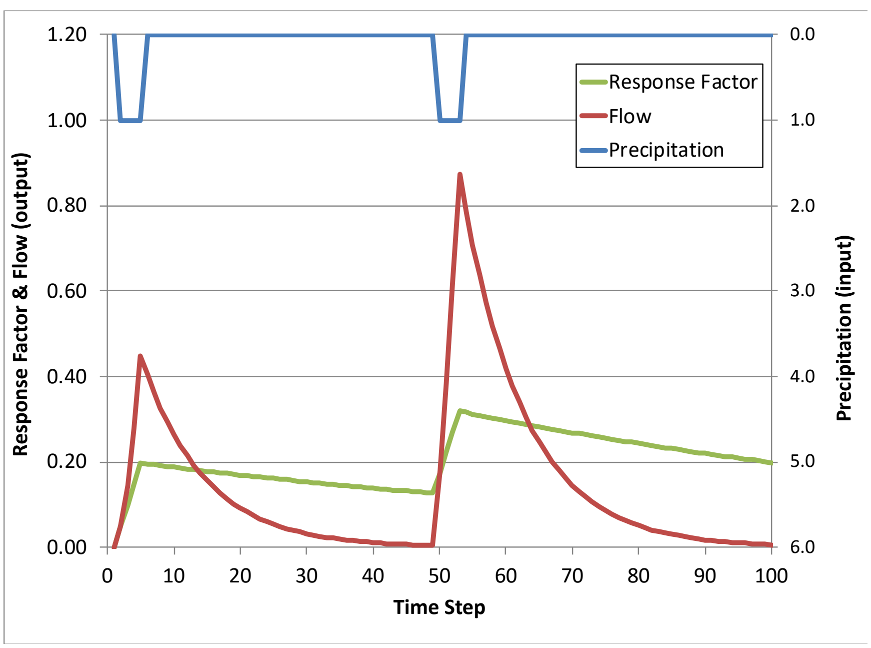

Figure 3 Antecedent Moisture Retention transfer function output from Equations 2 and 3 with back-to-back storms. SF = 0.9, AMRF = 0.99, TF = 0.05. Precipitation inputs of a value of 1 in the first four time steps, and four more precipitation inputs of a value of 1 starting in the 50th time step. Note the varying Response Factor is much higher for the second storm, as it has not fully recovered from the first storm, and this drives the flow up for the second storm.

3.4 Affine Constant

The antecedent moisture retention transfer function depicted in Equation 3 can cause the Response Factor to approach zero after a long period with no precipitation. However, many systems exhibit a non-zero lower bounds to the Response Factor, regardless of how dry the antecedent moisture conditions are. This might be the case for a system with some amount of directly connected impervious area, for example, which would be invariant to changes in antecedent moisture.

To address this, Equation 1 can be modified, as shown in Equation 4, to include an affine constant that is added to the Response Factor. This affine constant has the effect of placing a floor or lower bounds to the Response Factor that is used to compute flow.

$$Q_t = (AC + RF_t) * P_t + SF * Q_{t-1} \tag{4}$$

where:

AC = Affine Constant that is added to the Response Factor

3.5 Antecedent Moisture Precipitation Delay

The antecedent moisture retention transfer function depicted in Equation 3 uses the precipitation at the current time step (P_t) as the input. This was done for simplicity to demonstrate the form and function of the equations. However, in practice, this allows the precipitation to change the Response Factor output of Equation 3 at the same time step, essentially allowing precipitation to instantaneously modify the antecedent moisture level of the model, which does not make physical sense and can be problematic in modeling some systems.

To address this, Equation 3 can be modified, as shown in Equation 5, to use the precipitation at the previous time step (P_{t-1}). This prevents the current precipitation from changing the wetness level computation at the same time step in the rainfall-runoff linear transfer function in Equation 4.

$$RF_t = TF * P_{t-1} + AMRF * RF_{t-1} \tag{5}$$

where:

P_{t-1} = Precipitation input at the previous time step, *t-1*

3.6 Temperature Factor Transfer Functions

Linear Temperature Transfer Function

Hydrologic conditions such as air temperature, soil moisture, ground water levels, evaporation, evapotranspiration, solar radiation, and other factors all change from season to season and impact antecedent moisture conditions. This in turn and changes the system response continuously during storm events and in between storm events. Although these effects can be complex, air temperature closely mirrors the variations of many hydrologic conditions and has been found through regression analysis to have a significant relationship to volumetric capture variations due to antecedent moisture conditions (Zhang et al., 2011). Air temperature is also a fairly easy variable to obtain continuous measurements.

For modeling rainfall-runoff dynamics, air temperature can be used as a surrogate variable to represent these various hydrologic conditions. This can be done by continuously varying the Temperature Factor (TF) in Equation 5 based on the preceding air temperature. This is an inverse relationship, where low temperatures drive up the Temperature Factor and high temperatures drive down the Temperature Factor.

The reason for this inverse relationship is that lower air temperatures correspond with other hydrologic phenomenon that are more conducive to retaining antecedent moisture and result in higher wetness conditions, therefore resulting in higher system responses during rain events. For example, lower temperatures cause a slower rate of evaporation, which in turn increases wetness conditions. Lower temperatures also correspond to periods of lower solar radiation and lower tree and plant cover, both of which decrease rates of evaporation and evapotranspiration, which increase wetness conditions. These and other hydrologic dynamics correspond to air temperature, making this an excellent and simple surrogate variable for hydrologic modeling of seasonal antecedent moisture dynamics. By continuously varying the Temperature Factor based on air temperature, the response factor in Equation 5 will continuously change based on season conditions.

The antecedent moisture retention transfer function is then represented as shown in Equation 6. In this equation, unlike Equation 5, the value of the Temperature Factor (TF) changes with time. Application to many systems has shown that antecedent moisture conditions do not affect the Antecedent Moisture Retention Factor (AMRF) in Equation 6, suggesting that this might be a characteristic of the physical system (such as soil type, land use type, etc.), as opposed to the more temporal system changes (e.g. rainfall, air temperature, solar radiation, etc.)

$$RF_t = TF_t * P_{t-1} + AMRF * RF_{t-1} \tag{6}$$

where:

TF_t = Varying Temperature Factor at current time step, t

The continuously varying Temperature Factor (TF_t) from equation 6 can be represented as a simple linear transfer function, as shown in Equation 7. The equation has a single input of a Moving Average Temperature time series, and single output of the varying temperature factor, and two parameters, as shown in Equation 7.

$$TF_t = m * MAT_t + b \tag{7}$$

where:

TF_t = Varying Temperature Factor output at the current time step, t
MAT_t = Moving Average Temperature at time step, t, as defined in Equation 8 below.
m = Slope of inverse linear relationship
b = Intercept of inverse linear relationship

The temperature input time series is passed through an averaging function to smooth the temperature time series as shown in Equation 8. The air temperature is averaged between time steps t and t-n. This is done because air temperatures can change suddenly due to weather systems and it fluctuates diurnally, but seasonal antecedent moisture conditions tend to change more slowly due to longer-term temperature conditions. Experience has shown that a 10-15 day averaging period tends to provide an appropriately smoothed Moving Average Temperature.

$$MAT_t = \frac{1}{x} \sum_{n=0}^{x} T_{t-n} \tag{8}$$

where:

MAT_t = Moving Average Temperature at time step, t
x = Number of time steps to average for the temperature
T_{t-n} = Air temperature at the time step of t-n, where n varies from 0 to x.

Some notes on the Temperature Factor transfer function:

- The output from Equation 7 is a time series of how the Temperature Factor varies in time. It is similar in form to the hydrograph output from Equation 1 and the response factor time series output from Equation 5, but the temperature factor time series output represents how seasonal affects (represented by the air temperature surrogate variable) impact antecedent moisture conditions over time and affect the response factor and hydrograph response to rainfall.
- The terms m and b define a line that describes the relationship between air temperature and the Temperature Factor (TF_t).
- The slope (m) of the linear relationship is negative so that there is an inverse relationship between temperature and the Temperature Factor (TF_t).
- Continuously varying the Temperature Factor in Equation 6 based on the output from Equation 7 makes Equation 6 nonlinear in parameters.
- The linear representation in Equation 7 could cause the Temperature Factor output to become negative, which does not make physical sense. To prevent this, some bounds could be placed on the Temperature Factor output. This is discussed in more detail in later sections.

Figure 4 shows an example of the line that defines the relationship between the Temperature Factor and the temperature in Equation 7. This figure shows how example slope (m) and intercept (b) parameters relate to the line.

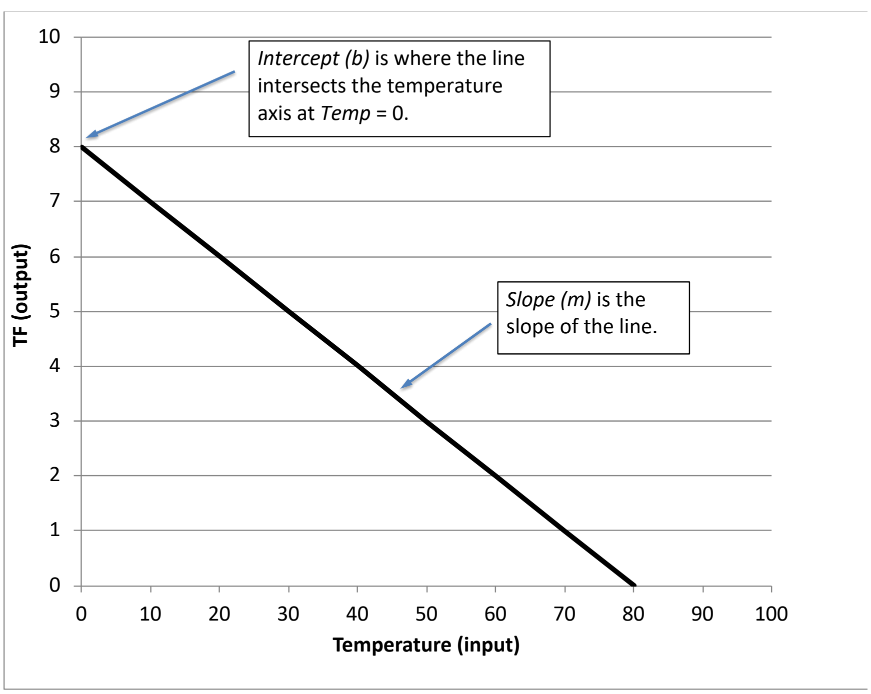

Figure 4 Example plot showing the relationship between temperature and Temperature Factor with the linear relationship defined by an example slope and an intercept. *m = -0.1, b = 8*.

Note that this is an inverse relationship where low temperatures yield high Temperature Factors.

Linear Temperature Factor Transfer Function Represented as Two Points

Creating a model of the temperature factor transfer function using the slope (*m*) and intercept *(b)* from Equation 7 can be hard to conceptualize because it is not intuitive how the *m* and *b* parameters affect the line. It can be more intuitive for modelers to define the line by two points on the line, instead of the slope (*m*) and intercept *(b)*. Figure 5 shows an example of these two points for the same linear relationship from Figure 4. The points are each represented as pairing of a *temperature* and a *temperature factor*.

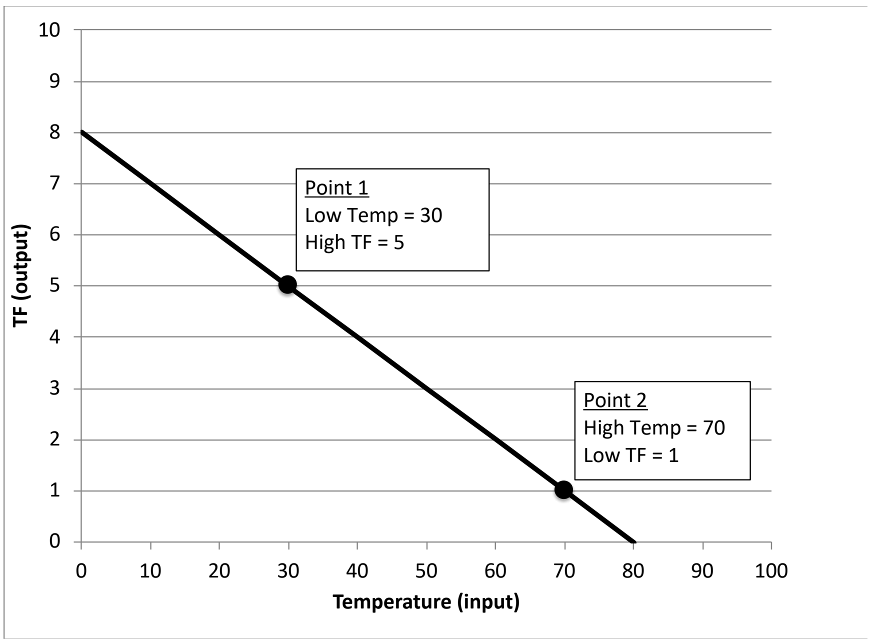

Figure 5 Plot showing the temperature versus the Temperature Factor with the linear relationship defined by two points on the line. Note the line has the same slope (m = -0.1) and an intercept (b = 8) as that shown in Figure 4.

The representation of the example line as shown in Figure 5 with two points allows the modeler to conceptualize the linear relationship as a TF value at a low temperature, and a TF value at a high temperature. This is more intuitive and easier to conceptualize when developing a model than with a slope (m) and an intercept (b). The slope (m) and intercept (b) can then be represented as shown in Equations 9 and 10.

$$m = \frac{(Low\ TF - High\ TF)}{(High\ Temp - Low\ Temp)} \tag{9}$$

$$b = Low\ TF - m * High\ Temp \tag{10}$$

where:

$Low\ TF$	=	Lower TF Value from Point 2
$High\ TF$	=	Higher TF Value from Point 1
$Low\ Temp$	=	Lower Temperature Value from Point 1
$High\ Temp$	=	Higher Temperature Value from Point 2

Substituting these into Equation 7 then yields Equation 11 for the temperature factor transfer function:

$$TF_t = \frac{(Low\ TF - High\ TF)}{(High\ Temp - Low\ Temp)} * MAT_t + Low\ TF - \frac{(Low\ TF - High\ TF)}{(High\ Temp - Low\ Temp)} * High\ Temp \quad (11)$$

where:

TF_t	=	Varying Temperature Factor output at the current time step, t
x	=	Number of time steps to average for the temperature
T_{t-n}	=	Air temperature at the time step of t-n, where n varies from 0 to x.

Some notes on Equation 11:

- Larger values for Low TF and High TF yield a larger temperature factor output time series from Equation 11. Because the temperature factor transfer function is linear, superposition applies, and the temperature factor output time series will scale proportional to the Low TF and High TF. For example, doubling the Low TF and High TF will double all points of the temperature factor output time series.
- For convenience, the two points on the linear relationship shown in Figure 9 can be represented by the nomenclature of TF Line: (Low Temp, High TF) and (High Temp, Low TF). For example, the temperature factor parameter shown in Figure 4 are then represented as: TF Line: (30,5) and (70,1).
- The values used for the Low Temp and High Temp are arbitrary. Values can be selected that are convenient to conceptualize the relationship between the temperature factor and temperature. This then allows the modeler to adjust the Low TF and High TF parameters to calibrate the model to fit observed data.

Figure 6 shows the results of the temperature smoothing with the Moving Average Temperature function of Equation 8 with some varying temperature data. Figure 7 shows an example Response Factor and flow output from Equations 5 and 6 with a varying Temperature Factor from Equation 11. Note that these temperature effects in nature normally occur in different seasons. The example in Figure 7 shows large temperature changes in a short time period to illustrate the mathematics.

The Response Factor in Figure 7 completely recovers after the first storm, but the Response Factor is much smaller for the second storm because the second storm occurs during high temperatures, which drives the Response Factor down. This in turn drives down the flow response for the second storm.

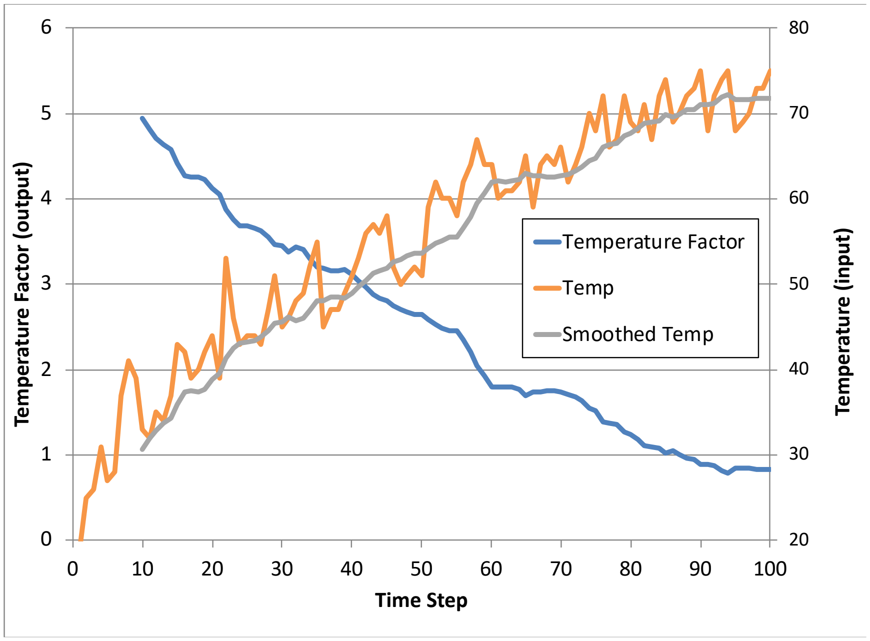

Figure 6 Temperature smoothing results from Equation 8 and Temperature Factor output from Equation 7, with temperature data varying and generally trending upward during period. Temperature averaged for ten time-steps (x = 10), TF Line: (30,5) and (70,1).

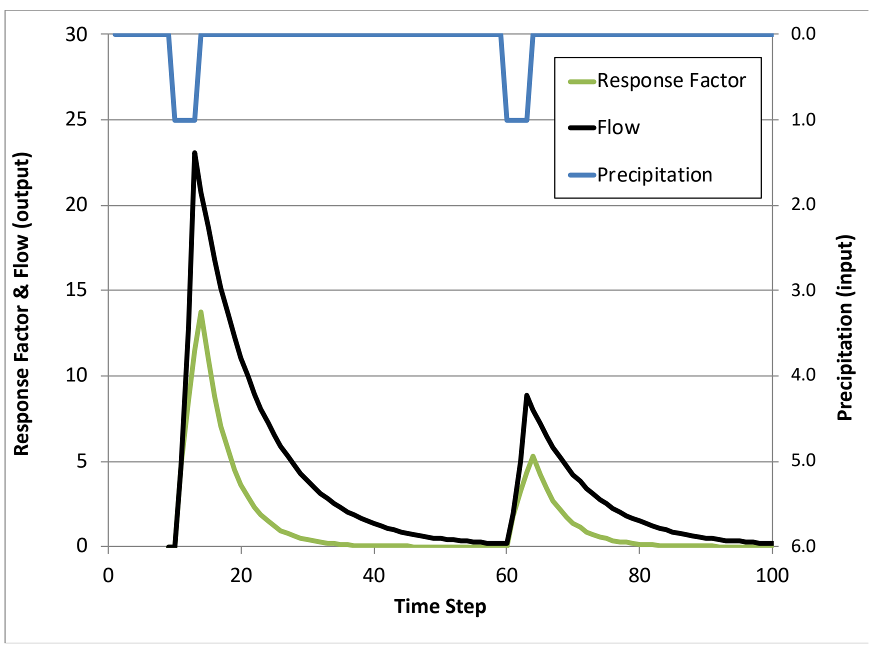

Figure 7 Output from Flow and Response Factor transfer functions from Equations 2 and 6 with the same precipitation as Figure 2, with temperature from Figure 6, and a variable Temperature Factor. Temperature averaged for ten time-steps (x = 10), SF = 0.9, AMRF = 0.8, TF Line: (30,5) and (70,1). The response factor and flow for the second storm are both smaller because of the higher temperatures during the storm and the results of the inverse relationship between TF and Temperature in Equation 7.

Sigmoid Temperature Factor Transfer Function

As noted above, the linear representation of the Temperature Factor variation in Equation 5 could cause the Temperature Factor output to become negative at very high temperature, which does not make physical sense. Conversely, at very low temperatures, the linear representation could cause the Temperature Factor output to become very high, which may not reflect the actual behavior of real systems.

To address this, a sigmoid function can be used. The sigmoid function results in an S-shaped curve that is nearly linear in the center, and very quickly asymptotically approaches set values near the limits of the function (von Seggern 2007). The characteristic of a sigmoid function makes it an excellent function for limiting the ranges of the temperature factor.

The temperature factor transfer function in Equation 7 can be replaced with a sigmoid function as shown in Equation 12.

$$TF_t = \left[\frac{L}{1+e^{(-k\{MAT_t - x_O\})}}\right] + High\,TF - \frac{11}{12}L \qquad (12)$$

where:

$$L = 1.2 \times (High\,TF - Low\,TF) \qquad (13)$$

$$k = \left[\frac{4.7964}{(Low\,T - High\,T)}\right] \qquad (14)$$

$$x_0 = \frac{(Low\,T + High\,T)}{2} \qquad (15)$$

Some notes on Equation 12 - 15:

- Like Equation 7, the output from Equation 12 is a time series of how the Temperature Factor varies in time. However, the use of a sigmoid function, rather than a linear function, results in some useful bounds on the ranges of Temperature Factors that can be generated.
- The constants used in the sigmoid function above cause the Temperature Factor output to be bounded by a range that is 20% larger than the range of the Low TF and the High TF. Other constants could be used if a different range were desired.
- For convenience, the two points on the sigmoid relationship can be represented by the nomenclature of TF Sigmoid: (Low Temp, High TF) and (High Temp, Low TF).

Figure 8 shows an example of the relationship between the temperature and the Temperature Factor with the sigmoid function in Equation 12.

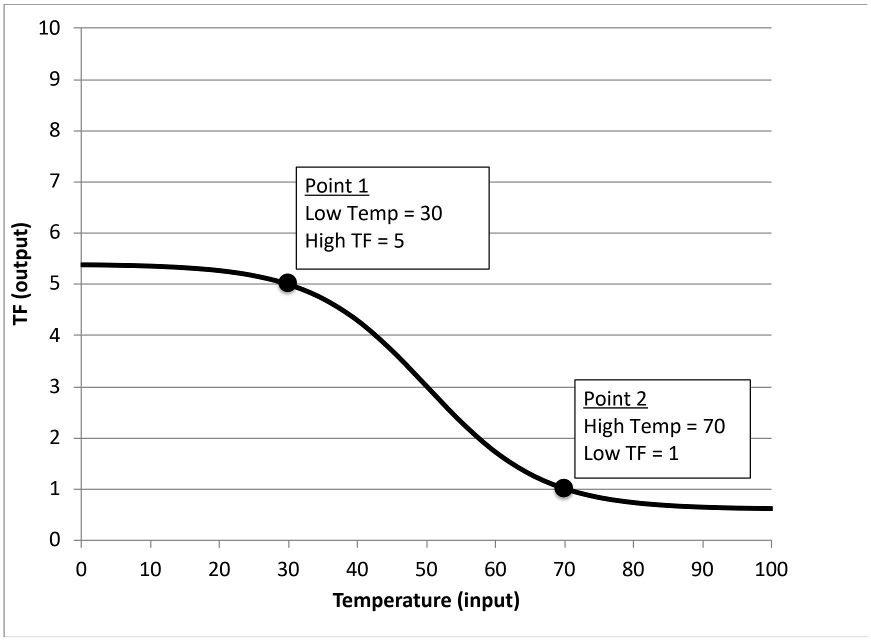

Figure 8 Plot showing the temperature versus the Temperature Factor with the sigmoid function defined by two points on the function. TF Sigmoid: (30,5) and (70,1).

3.7 Precipitation Smoothing Function

Smoothing the precipitation input by averaging several measurements before using it for input into the model can be useful for some systems. This is most common for systems with very long travel times or very long response times where the peak of the hydrograph shape is smoothed by attenuation, such as base flow, which is described in the next section. Precipitation smoothing can be achieved by substituting a moving average precipitation for the precipitation (P_t) in Equation 1 or 4. The moving average precipitation function is shown in Equation 16. Figure 9 shows the effects of the precipitation smoothing on the rainfall and flow from the linear rainfall-runoff transfer function in Equation 1.

$$MAP_t = \frac{1}{x} \sum_{n=0}^{x} P_{t-n} \tag{16}$$

where:

MAP_t = Moving Average Precipitation at time step, t

x = Number of time steps to average for the precipitation

P_{t-n} = Precipitation at the time step of t-n, where n varies from 0 to *x*.

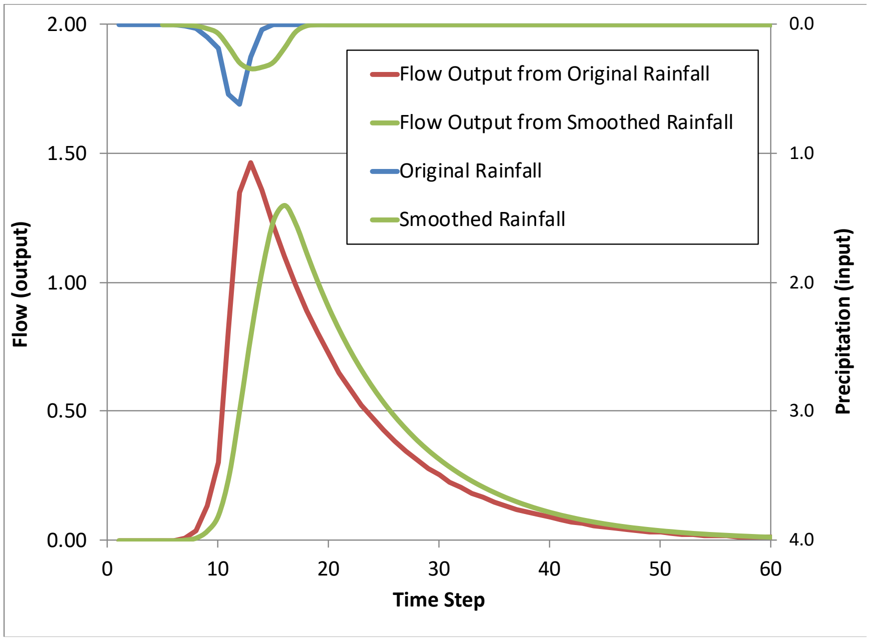

Figure 9 Plot showing the effects of rainfall smoothing on the linear rainfall-runoff transfer function in Equation 1 for an example precipitation. *SF = 0.9, RF = 1, Precipitation averaged for 5 time-steps (x=5).*

3.8 Restatement of Antecedent Moisture Transfer Functions

The previous sections show the derivation of the equations in order to step through their development and demonstrate their application to modeling rainfall-runoff responses and antecedent moisture effects. Figure 10 shows an infographic of the model equations and how they function to restate the final equations and parameter definitions in their final form for convenient reference.

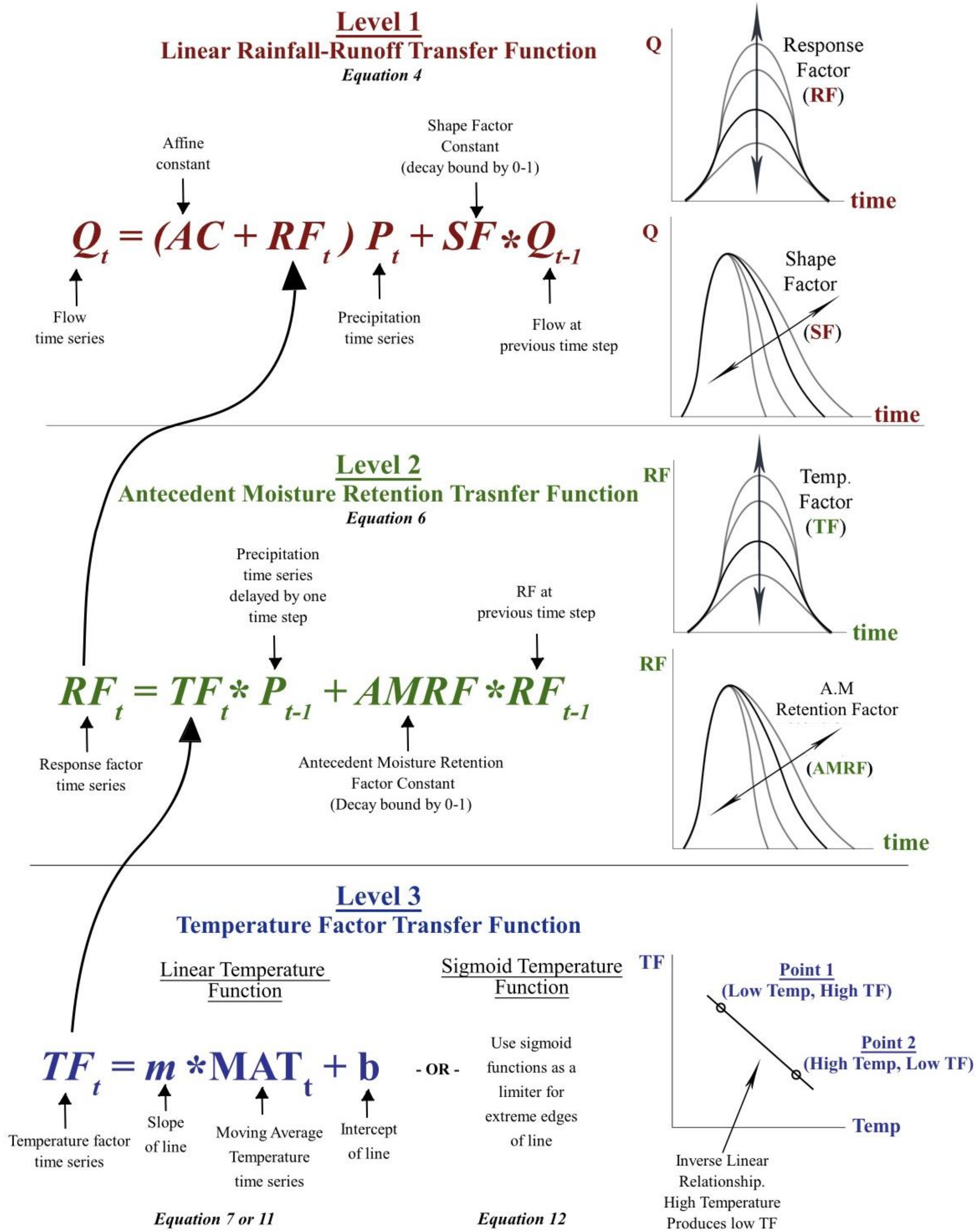

Figure 10- Infographic of Antecedent Moisture Model equations and process.

3.9 Base Flow Transfer Function

Some systems exhibit a base flow component that varies very slowly with a very long response time that can take weeks or months to react to climatological variations like precipitation and air temperature. In natural channels like streams and rivers this base flow component can be caused by ground water discharge into the channel. In sanitary collection systems this base flow component can be caused by Ground Water Infiltration (GWI) into the system.

This base flow component can be represented with a series of transfer functions that are similar in form to those described earlier, with some modifications. For example, the flow can be represented by the linear rainfall-runoff transfer function shown in Equation 4. Using this equation for base flow usually requires rainfall smoothing of several days or weeks with Equation 16 to achieve the observed base flow shape in most systems.

However, the antecedent moisture retention function in Equation 5 is not necessary for base flow modeling. The long response times for base flow result in relatively high values for the Shape Factor (> 0.99 for hourly time steps), which causes the flow output for one precipitation event to continue for a long period of time and effect subsequent precipitation events. This effect addresses the antecedent moisture impacts from back to back precipitation events sufficiently for base flow, such that the antecedent moisture retention transfer function shown in Equation 5 is not necessary for modeling base flow. This simplifies the modeling process and makes it more parsimonious by reducing the modeling parameters.

Additionally, some systems require the addition of a base flow constant to the flow output to match system observations. This may be caused by a constant discharge to the system or a flow component like GWI that has a lower bound.

Base flow can then be represented as linear rainfall-runoff transfer function as shown in Equation 17, which includes both a varying base flow component and a constant base flow component *(BF).* The varying Response Factor (RF_t) in Equation 17 can then be represented as shown in Equation 18. This equation has a similar form as Equation 7 to use the linear temperature factor transfer function to directly modify the Response Factor (RF_t). Similarly, the same forms of Equations 11 and 12 can be used to define the Response Factor transfer function for base flow as two points or as a sigmoid function, respectively.

$$Q_t = (AC + TF_t) * MAP_t + SF * (Q_{t-1} - BF) + BF \tag{17}$$

$$TF_t = m * MAT_t + b \tag{18}$$

where:

BF	=	Constant base flow

Figure 11 is a plot that shows the output from the base flow model. The model has been calibrated to real observations from a separate sanitary collection system to give the output some context. Note that the model simulates the very long response times and the greatly dampened rainfall signal that influences GWI, overcoming the limitations of black-box regression methods (Wright et al., 2001).

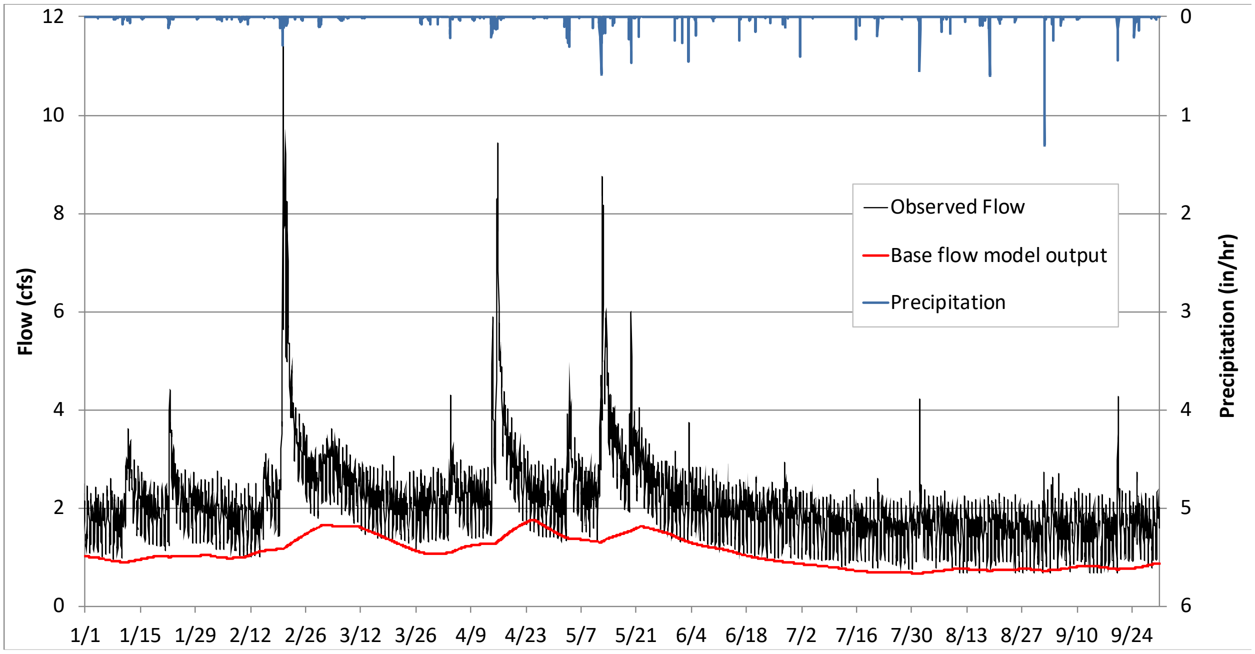

Figure 11 Output flow from the base flow transfer functions from Equations 17, with the real system observations for reference. Data and model at hourly time steps, temperature and precipitation averaged for ten days (*x = 240), BF = 0.45, SF = 0.9982, AC = 0.03, TF Sigmoid: (30,0.38) and (70,0.10).* The base flow model was calibrated to match the daily minimum value of the diurnals flow pattern, which is a good indicator of GWI flow. The difference between the observed and modeled flows is comprised of the diurnal pattern, inflow and infiltration flow components, which can be modeled separately.

4 Physical Interpretation of the Antecedent Moisture Model

A grey box model combines a partial theoretical structure with data-based approaches to complete the model (Bohlin). The antecedent moisture model described here is a grey box model because portions of the model were selected based on physical insights of the system, and the model components relate to physical processes in the system. Physical interpretation of the model structure and parameters is possible as a result. This overcomes several disadvantages of "black box" regression methods and allows engineers to gain useful insights into the system (Wright et al., 2001).

Depending on the system being modeled, several combinations of rainfall-runoff transfer functions may be necessary (base flow, slow, fast, etc.). For example, in sanitary collection systems, the total flow response to a precipitation event can be subdivided into the components of base sanitary flow, GWI, inflow and infiltration (Dent et al., 2000). These flow components can be modeled separately and added together to derive the total system flow, as shown in Equation 19 and depicted in Figure 12.

$$Total\ Q_t = Inflow\ Q_t + Infiltration\ Q_t + \ Base\ Flow\ Q_t \qquad (19)$$

where:

$Total\ Q_t$	=	Total flow rate output at the current time step, *t*
$Inflow\ Q_t$	=	Inflow (Fast) flow rate output at the current time step, *t*
$Infiltration\ Q_t$	=	Infiltration (slow) flow rate output at the current time step, *t*

$Base\ Flow\ Q_t$ = Base flow GWI (very slow) flow rate output at the current time step, t

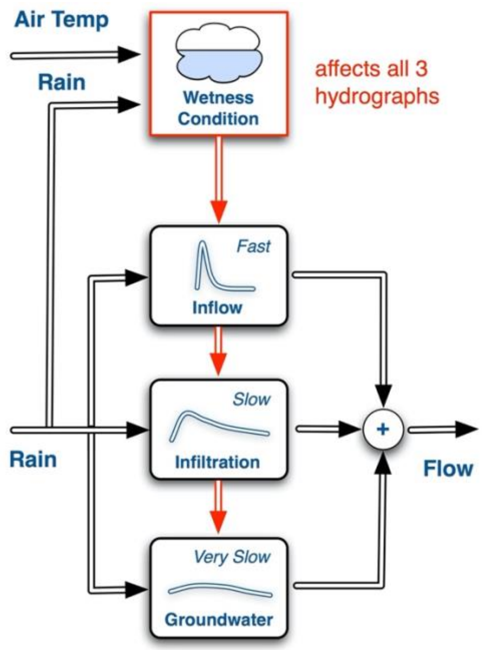

Figure 12 Schematic representation of the antecedent moisture model for sanitary flow components present in sanitary collection systems.

This physical representation of the model components allows engineers to identify and individually simulate the relative contributions of various flow components like those shown in Figure 12. These can be used by engineers to match system improvements and source removal techniques to the specific flow components present, and to estimate the impact on the system hydrograph from these improvements.

As previously noted, the rainfall-runoff linear transfer function in Equation 1 has the same form as the equation to represent an instantaneous unit hydrograph (IUH) (Bedient et al., 2019) by setting RF + SF = 1 in Equation 1. In fact, any values for the shape factor (SF = a) and Response Factor (RF = b) can be represented as a scaled IUH in Equation 1 by setting RF = 1 – a, to derive a IUH, and then scaling the output of the IUH by b / (1 - a). Furthermore, the Response Factor (RF) in Equation 1 is directly proportional to the volume, peak flow and value of every point of the output flow hydrograph (superposition of a linear equation). For example, the volume of the system response from Equation 1 can be computed as shown in Equation 20.

$$V = \sum P_t \ * \frac{RF}{(1-SF)} \tag{20}$$

where:

V = Volume of runoff hydrograph
$\sum P_t$ = Sum of the precipitation input time series

The volume of the observed hydrograph and resulting capture percentage (i.e. the percentage of rainfall captured by the runoff process) are the primary measures that continuously change in systems due to antecedent moisture conditions. The form of Equation 20 means that the value of the Response Factor (RF) in Equation 1 is proportional to the runoff volume and therefore the capture percentage of the system. This means that the predicted value of the Response Factor (RF_t) in Equation 5 represents how the capture percentage varies continuously in time. Therefore, the continuously varying Response Factor (RF_t) in Equation 5 represents the general wetness condition of the district being modeled in continuous time.

When using Equation 1 to simulate the rainfall-runoff response, changes to the system contributing area, such as that from a sewer separation program, can be represented as proportional changes to the RF parameter. For example, half of the RF would generate half of the flow and volume, representing half of the contributing area. When modeling antecedent moisture contributions using Equations 4, 5 and 12, changes in contributing area can be represented as proportional changes to the AC, Low TF and High TF parameters. Proportionally changing these three parameters will also proportionally change the volume, peak flow and every flow point of the output hydrograph. This is useful in relating the model parameters to a physical characteristic of the system like drainage area.

The Shape Factor (SF) in Equations 4 represent the rate of recession (decay) of the hydrograph after the precipitation has stopped. The observation by the Author that this parameter does not appear to change due to variations in antecedent moisture conditions as conceptualized in these proposed transfer functions indicates that this parameter might be a characteristic of the physical system (such as district shape, slope, travel time, etc.) This invariance allows antecedent moisture conditions due to preceding precipitation to be explicitly identified and modeled through variations of the Response Factor (RF).

The Antecedent Moisture Retention Factor (AMRF) in Equations 5 represents the rate of recession (decay) of wetness conditions after the precipitation has stopped. The observation by the Author that this parameter does not appear to change due to variations in antecedent moisture conditions as conceptualized in these proposed transfer functions indicates that this parameter might be a characteristic of the physical system (such as soil type, land use type, etc.). This invariance allows antecedent moisture conditions due to season variations to be explicitly identified and modeled through variations of the Temperature Factor (TF). The TF High and TF Low parameters in Equation 12 describe how much rainfall increases the wetness conditions.

5 Conclusion

The equations and process were presented for a relatively simple model that accurately simulates the impacts of antecedent moisture conditions on the rainfall-runoff process. The model is a non-linear, lumped, conceptual, deterministic, grey-box model developed from the principles of system identification and the parsimony principle. The model contains relatively few parameters, which has several advantages in modeling, including ease of use, fewer parameters to calibrate, ability to quickly identify optimal parameters, and ease to represent in a numerical computer routine, and the ability to make accurate predictions of unobserved or design conditions, which is a critical function of a model. Physical interpretation of the model structure and parameters is possible, providing the modeler with useful insights into the physical processes driving the rainfall-runoff dynamics.

The model was developed primarily from observations in the Midwest U.S. where both preceding rainfall and seasonal effects (represented by air temperature in the model) impact antecedent moisture conditions. In other climates, different input observations and a different model structure may be necessary to simulate antecedent moisture dynamics. Others are encouraged to examine the equations presented here, and modify their structure, inputs and parameterization to identify the simplest and most accurate model for their application.

The author is currently developing a manual of practice to accompany the equations outlined here that will cover other important aspect of the antecedent moisture modeling process. Please contact the author if you are interested in corresponding on this. Topics include:

- Spreadsheet companion to the equations.
- Data requirements and observation data content for developing a model.
- Diurnal flow filtering from total flow signal in sewer applications.
- Storm events selection for model calibration.
- Process and steps for calibrating the model.
- Methods for model validation.
- Model performance quantification through a rigorous accuracy of fit process.
- Long-term continuous simulation and frequency analysis for system design.
- System benchmarking under identically simulated antecedent moisture conditions using the model.

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